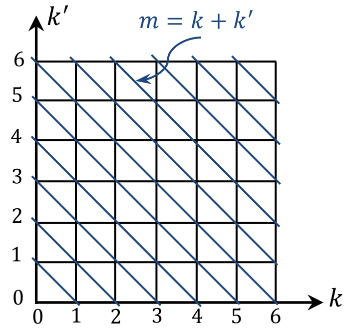
Problem 18) The binomial expansions of $(x+y)^n$ and $(x+y)^{2n}$ are written straightforwardly, as follows:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \tag{1}$$

$$(x+y)^{2n} = \sum_{m=0}^{2n} \binom{2n}{m} x^{2n-m} y^m. \tag{2}$$



Squaring Eq.(1) now yields

$$(x+y)^{2n} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \times \sum_{k'=0}^{n} \binom{n}{k'} x^{n-k'} y^{k'}$$

$\boxed{k + k' = m} \rightarrow$

$$= \sum_{k=0}^{n} \sum_{k'=0}^{n} \binom{n}{k} \binom{n}{k'} x^{2n-k-k'} y^{k+k'}$$

$$= \sum_{m=0}^{2n} \left[\sum_{k=\max(0,m-n)}^{\min(m,n)} \binom{n}{k} \binom{n}{m-k} \right] x^{2n-m} y^m. \tag{3}$$

At $m = n$, the coefficient of $x^{2n-m} y^m$ in Eq.(2) is $\binom{2n}{n}$. The corresponding coefficient in Eq.(3) is $\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}$. Thus, considering that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$, we will have

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{4}$$